On a Combinatorial Identity of Djakov and Mityagin

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Abstract

Consider a pyramid made out of unit cubes arranged in square horizontal layers, with a ledge of one cube's length around the perimeter of each layer. For any natural number k, we can count the number of ways of choosing k unit cubes from the pyramid such that no two cubes are in the same horizontal layer; we can also count the number of ways of choosing k unit cubes from the pyramid such that no two cubes come from the same vertical slice (taken parallel to a fixed edge of the pyramid) *or* from two adjacent slices. Djakov and Mityagin first established, using functional analysis, that these two quantities are always equal (the enumerative interpretation given here is due to Thomas Kalinowski). Don Zagier supplied the first combinatorial proof of this result. We provide a new, more natural combinatorial proof.

The authors of [3] conjectured ([1, Conjecture 1]; [2, 3, Theorem 4]) that the following identity holds for all positive integers n and integers k:

$$\sum_{J\in\binom{[n]}{k}^*}\prod_{j\in J} j^2 = \sum_{J\in\binom{[n]}{k}^{**}}\prod_{j\in J} j(n+1-j) \tag{1}$$

where $[n] := \{1,2,3,\ldots,n\}$ as usual, $\binom{[n]}{k}^*$ denotes the collection of k-subsets of $[n]$ all of whose elements are congruent to n modulo 2, and $\binom{[n]}{k}^{**}$ denotes

the collection of k-subsets of $[n]$ which do not contain two consecutive integers. Don Zagier gave a combinatorial proof of this result in the appendix to [2, 3]. It turned out that a few years prior to this, the same identity had been established, with a less direct, non-combinatorial proof, in the context of functional analysis ([4, 5]). In the present note, we establish the identity in the most straightforward fashion yet, by calculating the characteristic polynomial of a certain matrix in two different ways.

Given two sequences (a_j) and (b_j) of real numbers, consider the $(n+1)$ by $(n+1)$ matrix

$$M_n = \begin{bmatrix} 0 & b_1 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & 0 & b_2 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 0 & b_3 & \cdots & 0 & 0 \\ 0 & 0 & a_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_n \\ 0 & 0 & 0 & 0 & \cdots & a_n & 0 \end{bmatrix}, \tag{2}$$

where the entries in the diagonal below the main diagonal are $a_1, a_2, \ldots, a_n$, the entries in the diagonal above the main diagonal are $b_1, b_2, \ldots, b_n$, and the remaining entries are 0. Let $\chi_n(x) = \det(xI - M)$ be the characteristic polynomial of M_n in the variable x. We claim that for all $n \geq 1$ and all $k \in \mathbf{Z}$, the coefficient $d_{k,n}$ of x^{n+1-2k} in $\chi_n(x)$ satisfies

$$d_{k,n} = (-1)^k \sum_{J \in \binom{[n]}{k}^{**}} \prod_{j \in J} a_j b_j. \tag{3}$$

One way to see this is by observing that both sides of Equation 3 satisfy the recurrence

$$r_{k,n} = r_{k,n-1} - a_n b_n r_{k-1,n-2} \tag{4}$$

for $n \geq 3$, and they agree for $n = 1, 2$. For the left-hand side of Equation 3, the recursion follows by expanding the determinant about the last row. For the right-hand side, it follows by considering the terms of total degree k with respect to the variables $x_1, x_2, \ldots, x_n$ in the product

$$\prod_{j=1}^{n} (1 - x_j a_j b_j), \tag{5}$$

after eliminating terms in which two consecutive indices appear.

Specializing to the case $a_i = i$ and $b_i = n + 1 - i$, the matrix M_n becomes what Taussky and Todd term the "Kac matrix" S_n (see [6] for the long and impressive history of this matrix[1]; though named for Victor Kac, the notation honors Sylvester, who found its characteristic polynomial in the 1850s). We only need the eigenvalues of S_n; these are computed, e.g., in [6], but we present an original derivation of the spectrum in what follows.

A non-zero vector $v = [v_1\ v_2\ \ldots\ v_{n+1}]^{\mathrm{t}}$ is an eigenvector of S_n with eigenvalue λ if and only if the components v_i of v satisfy the relations

$$iv_i + (n-i)v_{i+2} = \lambda v_{i+1} \tag{6}$$

for all $i \in \{0, 1, 2, \ldots, n\}$; notice that the coefficients of v_0 and v_{n+2} will be 0, so this really does work.

Let d be an integer between 0 and n (inclusive). Set $v_1 = 1$ (arbitrarily), and use Equation 6 to obtain $v_2, \ldots, v_{d+1}$ by setting

$$v_{i+2} = \frac{(n-2d)v_{i+1} - iv_i}{n-i} \tag{7}$$

for $i = 0, 1, \ldots, d-1$. Let $p(x)$ be the unique polynomial of degree at most d with values $p(i) = v_i$ for $i \in \{1, 2, \ldots, d+1\}$. Then the equation

$$xp(x) + (n-x)p(x+2) = \lambda p(x+1) \tag{8}$$

holds, with $\lambda = n - 2d$, for at least the d distinct values $0, 1, 2, \ldots, d-1$ of x. Now both sides of Equation 8 are polynomials of degree at most d. If $\deg(p) < d$, then both sides have degree less than d, so Equation 8 is an identity for p. If $\deg(p) = d$, then both sides of Equation 8 not only have degree d, but also have the same leading coefficient of $(n-2d)c_d$, where c_d is the leading coefficient of p; subtracting this term from both sides, we find two polynomials of degree at most $d-1$ which agree at d distinct values of x, so again they are identically equal. It follows that $[p(1)\ p(2)\ \cdots\ p(n+1)]^{\mathrm{t}}$ is an eigenvector of S_n with eigenvalue $n - 2d$.

Since S_n is an $n+1$ by $n+1$ matrix, it follows that the characteristic polynomial of S_n is

$$\chi_n(x) = \prod_{d=0}^{n}(x - (n-2d)) = x^{\varepsilon} \cdot \prod_{\substack{1 \le j \le n \\ j \equiv n \pmod 2}} (x^2 - j^2), \tag{9}$$

[1]The author wishes to thank R. Brualdi for informing him of this earlier work.

where $\varepsilon = 0$ if $n \equiv 1 \pmod 2$ and $\varepsilon = 1$ if $n \equiv 0 \pmod 2$.

From Equation 9, we see that the coefficient of x^{n+1-2k} in $\chi_n(x)$ is

$$d_{k,n} = (-1)^k \sum_{J \in \binom{[n]}{k}^*} \prod_{j \in J} j^2. \tag{10}$$

Now Equation 1 follows by comparing Equations 3 and 10.

References

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